Pergamon

J. Appl. Maths Mechs, Vol. 57, No. 5, pp. 921–926, 1993

0021–8928/93 $24.00+0.00

0021–8928(93)E0032-R

AN IMPROVED THEORY OF THE VIBRATIONS OF A MULTILAYER ORTHOTROPIC PLATE†

S. V. KOLESNIKOV

St Petersburg

(*Received* 17 *November* 1992)

An improved theory of the vibrations of multilayer orthotropic plates of finite dimensions is proposed, based on the method of hypotheses [1, 2] and expansion of the normal displacements in powers of the plate thickness. The choice of the hypotheses is governed by the possibility both of using the results in practice and of solving new problems in the theory of the vibrations of an elastic rectangle.

CONSIDER a cylindrical stiff plate consisting of an odd number $(2m+1)$ of orthotropic layers arranged symmetrically about the middle layer, to which we assign the index zero $(i=0)$. Layers above the middle one are assigned positive indices from $i=1$ to $i=m$, and layers below it, negative indices from $i=-1$ to $i=m$. Layers symmetrical about the middle one have the same thicknesses and elastic parameters. The directions of the cylindrical system of coordinates X, Y, Z are assumed to coincide with the principal anisotropic directions of the elastic material. The origin is situated at the half-thickness of the plate.

The main aim of this paper is to study the motion of the multilayer plate in the XOZ plane, which cuts across the cross-section of the cylinder and is perpendicular to the OY axis. Our derivation of the equations of motion will be based on a two-dimensional formulation of the problem and the following hypotheses:

1. the differential equations of equilibrium and the components of the strain for a weakly bent plate will be based on the assumption that the stressed state of the curved plate is identical with the corresponding state for a flat plate, i.e. the curvature of the plate may be ignored in the equations of motion;

2. the shear stress F_{xzi} varies with the thickness of the multilayer plate in accordance with a given law [1, p. 46]

$$F_{xzi} = \varphi(x,t)f(z), \qquad f(z) = f(-z), \qquad f(z)|_{z=\pm h} = 0$$
$$\varphi(x,t) \equiv \varphi, \qquad F_{xzi} \equiv F_{xz}$$

3. the normal displacement may be expanded in powers of the plate thickness

$$U_{zi} = ax + \sum_{j=0}^{2N+1} \left(\frac{z}{h}\right)^j W_j(x,t)$$
$$W_j(x,t) \equiv W_j, \qquad U_{zi} \equiv U_z$$

where x, z are curvilinear coordinates, t is the time, $2h$ is the thickness of the plate, F_{xzi} and U_{zi} are the shear stress and normal displacement in the ith layer, $f(z)$ is a function characterizing

†*Prikl. Mat. Mekh.* Vol. 57, No. 5, pp. 160–165, 1993.

the variation of the shear stress with plate thickness, φ is an unknown function, a is the unknown angle through which the plate rotates as a whole, W_j is the unknown component of the normal displacement in its expansion in powers of the plate thickness, and $2N+1$ is the number of expansion terms.

The equilibrium conditions for the material of the ith layer for harmonic variations in weakly curved cylindrical coordinates are given by the following differential equations [2, p. 18]

$$\begin{aligned} \partial\sigma_{xi}/\partial x + \partial F_{xzi}/\partial z + \omega^2\rho_i U_{xi} &= 0 \\ \partial\sigma_{zi}/\partial z + \partial F_{xzi}/\partial x + \omega^2\rho_i U_{zi} &= 0 \end{aligned} \tag{1}$$

where i is the layer index (i varies from $-m$ to m), ρ_i is the density of the layer material, U_{xi}, U_{zi} and σ_{xi}, σ_{zi} are the displacements and principal stresses in the layer in the direction indicated, F_{xzi} is the shear stress, and ω is the angular frequency.

Substituting the given expressions for the shear and normal stresses (evaluated at the middle surface of the displacement) into the second equilibrium condition (1), integrating the result with respect to the normal coordinate z from z_i to z (where z lies within the ith layer), and taking into account that the normal stresses at the layer faces must be equal, we find an expression for the normal component of the principal stress at any point of the multilayer plate

$$\begin{aligned} \sigma_{zi} &= \sigma - J_0(z)\partial\varphi/\partial x - \\ &- \omega^2\left[ax(RZ_{i0} + I_{i0}(z)) + \sum_{j=1}^{2N+1} W_j(I_{ij}(z) + RZ_{ij})\right] \end{aligned} \tag{2}$$

$$J_i(z) = \int_{z_i}^{z} f(z)dz, \qquad I_{ij}(z) = \rho_i \int_{z_i}^{z} \left(\frac{z}{h}\right)^j dz$$

$$RZ_{\pm s,\, j} = RZ_{\pm s\mp 1,\, j} + I_{\pm s\mp 1,\, j}(z_{\pm s}), \quad RZ_{0j} = 0, \quad Z_0 = 0$$

$$S = 1, 2, \ldots, m, \qquad j = 0, 1, \ldots, 2N+1$$

where σ is an unknown constant of integration, Z_1, Z_2, ..., Z_{m+1} are the coordinates of the upper faces of layers $0, 1, \ldots, -m$, and Z_{-1}, Z_{-2}, ..., Z_{-m-1} are those of the lower faces of layers $0, -1, \ldots, -m$.

Using (2), the fact that $f(z)$ is an even function and the symmetrical arrangement of the layers about the middle layer, we find an equation for the equilibrium of the plate and the constant of integration

$$\begin{aligned} \sigma_2 - \sigma_1 &= Q_{11}\partial\varphi/\partial x - 2\omega^2[axRZ_{m+1,0} + W_0RZ_{m+1,0} + W_2RZ_{m+1,2} + \ldots] \\ \sigma_2 + \sigma_1 &= 2\sigma - 2\omega^2[W_1RZ_{m+1,1} + W_3RZ_{m+1,3} + \ldots] \end{aligned} \tag{3}$$

where σ_1 and σ_2 are the normal stresses at the lower and upper faces of the plate.

Differentiating the first equation of (3) with respect to x, we obtain the first equation of motion of the plate

$$\begin{aligned} &\partial(\sigma_2 - \sigma_1)/\partial x = Q_{11}\partial^2\varphi/\partial x^2 + \omega^2(-aH_1 + P_{12}\alpha_0 + P_{13}\alpha_2 + P_{14}\alpha_4 + \ldots) \\ &\alpha_j = \partial W_j/\partial x, \quad W_j = C_j + \int \alpha_j\, dx, \quad P_{1s} = -2RZ_{m+1,2s-4} \\ &H_1 = 2RZ_{m+1,0}, \quad S = 2, 3, \ldots, N+2, \quad j = 0, 1, \ldots, 2N+1 \end{aligned} \tag{4}$$

where α_j is the unknown component of the angle of rotation of the plate in its expansion in powers of the thickness, and C_j are unknown constants of integration.

The stressed state of the orthotropic material of layer i is determined by the semi-inverse method of the theory of elasticity, using Hooke's law [1, pp. 21, 46]

$$\sigma_{xi} = B_i e_{xi} + A_i \sigma_{zi}, \qquad B_i = \frac{E_1^i}{1-\upsilon_{12}^i \upsilon_{21}^i}$$
$$F_{xzi} = G_i e_{xzi}, \qquad A_i = \frac{E_1^i}{E_3^i} \frac{\upsilon_{13}^i + \upsilon_{12}^i \upsilon_{23}^i}{1-\upsilon_{12}^i \upsilon_{21}^i} \tag{5}$$

where e_{xi}, e_{xzi} are the volume and shear strains in the ith layer, υ_{12}^i, υ_{13}^i, υ_{21}^i, υ_{23}^i are Poisson's ratios, E_1^i, E_3^i are Young's moduli and G_i is the shear modulus.

Let us rewrite the last equation of (5) in terms of weakly curved cylindrical coordinates [2, p. 18], in the form

$$\partial U_{xi} / \partial z + \partial U_{zi} / \partial x = G_i^{-1} f(z) \varphi(x,t)$$

Integrating this equation with respect to z from z_i to z (where z is within the ith layer) and taking into account that the displacements at the layer faces are equal, we obtain an expression for the tangential displacements at any point of the plate

$$U_{xi} = b + V - az - \sum_{j=0}^{2N+1} \frac{z}{(1+j)} \left(\frac{z}{h}\right)^j \alpha_j + \varphi(AI_i + J_i(z)/G_i) \tag{6}$$

$$V_{x0}|_{z=0} = b + V, \quad V(x,t) \equiv V, \quad e_{xi} = \partial U_{xi} / \partial x$$
$$AI_{\pm s} = AI_{\pm s\mp 1} + J_{\pm s\mp 1}(z_{\pm s}) / G_{\pm s \mp 1}, \quad AI_0 = 0$$
$$S = 1, 2, \dots, m+1$$

where b is an unknown constant, equal to the tangential displacement of the plate as a whole, and V is the tangential displacement of the middle surface due to the plate vibrations.

Substituting (6) into the first equation of (5), we obtain

$$\begin{aligned}\sigma_{xi} = {} & B_i \frac{\partial V}{\partial x} - B_i \left(\frac{z^2}{2h} \frac{\partial \alpha_1}{\partial x} + \frac{z^4}{4h^3} \frac{\partial \alpha_3}{\partial x} + \dots \right) + A_i \sigma - \\ & - \omega^2 A_i [W_1(RZ_{i1} + I_{i1}(z)) + W_3(RZ_{i3} + I_{i3}(z)) + \dots] - \\ & - B_i \left(z \frac{\partial \alpha_0}{\partial x} + \frac{z^3}{3h^2} \frac{\partial \alpha_2}{\partial x} + \dots \right) + \frac{\partial \varphi}{\partial x} \left[B_i \left(AI_i + \frac{J_i(z)}{G_i} \right) - A_i J_0(x) \right] - \\ & - \omega^2 A_i [ax(RZ_{i0} + I_{i0}(z)) + W_0(RZ_{i0} + I_{i0}(z)) + W_2(RZ_{i2} + I_{i2}(z)) + \dots]\end{aligned} \tag{7}$$

Using the first equation of equilibrium (5), Eqs (2)–(4), (7) and the fact that the moments acting over the cross-section of the plate are equivalent, we obtain the equations of the plate

$$\frac{\partial^2}{\partial x^2} DU + \omega^2 RU = T \frac{\partial}{\partial x} (\sigma_1 + \sigma_2) + \omega^2 Kb$$
$$\frac{\partial^2}{\partial x^2} QW + \omega^2 PW = \left\| \begin{matrix} \partial(\sigma_1 + \sigma_2)/\partial x \\ -4\varphi/3 \\ \vdots \\ -4\varphi(2N+3)^{-1} \end{matrix} \right\| + \omega^2 Ha \tag{8}$$

$$D = (D_{js}), \quad R = (R_{js}), \quad T = (T_j), \quad K = (K_j)$$

$$Q = (Q_{js}),\quad P = (P_{js}),\quad H = (H_j),\quad s, j = 1, 2, \dots, N+2$$

$$\begin{Vmatrix} M_0 \\ \vdots \\ M_{2N} \end{Vmatrix} = \frac{\partial}{\partial x} DU \qquad \begin{Vmatrix} M_1 \\ \vdots \\ M_{2N+1} \end{Vmatrix} = \frac{\partial}{\partial x} \begin{Vmatrix} Q_{21} & \cdots & Q_{2,N+2} \\ \vdots & & \vdots \\ Q_{N+2,1} & \cdots & Q_{N+2,N+2} \end{Vmatrix} W$$

$$U' = (V, \alpha_1, \dots, \alpha_{2N+1}),\qquad W' = (\varphi, \alpha_0, \dots, \alpha_{2N})$$

where U and W are column-matrix functions, U' and W' are transposed matrix functions, and $M_0, \dots, M_{2N+1}$ are the moments of the zeroth, first, second, etc., orders. The explicit form of the matrix coefficients will be given at the end of the paper.

The motions of the plate are described by two systems of matrix wave equations—symmetric and antisymmetric vibrations. It is obvious from (8) that the number of equations in each linear system of differential equations and the number of boundary conditions are determined by the appropriate terms in the expansion of the normal stress in series in powers of the plate thickness. If only the zeroth term of this series is considered, one obtains the usual equation for the symmetric vibrations, while the system of equations for the antisymmetric vibrations differs from the usual equations only in its notation and in the unknown functions that have to be determined [1, p. 238]. The difference is that the equations of motion and approximate boundary conditions are written in terms of the unknown angles of rotation of the plate, not in terms of the unknown displacements. The new notation for the equations of motion and the boundary conditions is essential in studying vibrations of thick multilayer plates of finite dimensions, since in that case the operators describing the homogeneous matrix equations have adjoints [3, p. 413; 4, p. 20].

The general solution of problem (8)' for plates of finite size may be sought as a series in column-matrix eigenfunctions [5, p. 78], where the latter satisfy the homogeneous wave equations of the natural vibrations of the plate

$$\begin{gathered} (L_s + \omega_n^1 R)U_n = 0,\qquad U_n' = (V_n, \alpha_{1n}, \dots, \alpha_{2N+1,n}) \\ (L_a + \varepsilon_l^2 P)W_l = 0,\qquad W_l' = (\varphi_l, \alpha_{0l}, \dots, \alpha_{2N,l}) \\ L_s = D\partial^2/\partial x^2,\quad L_a = Q\partial^2/\partial x^2,\quad O' = (0, \dots, 0) \\ \begin{Vmatrix} M_{0n} \\ \vdots \\ M_{2N,n} \end{Vmatrix} = \frac{\partial}{\partial x} DU_n \qquad \begin{Vmatrix} M_{1l} \\ \vdots \\ M_{2N+1,l} \end{Vmatrix} = \frac{\partial}{\partial x} \begin{Vmatrix} Q_{21} & \cdots & Q_{2,N+2} \\ \vdots & & \vdots \\ Q_{N+2,1} & \cdots & Q_{N+2,N+2} \end{Vmatrix} W_l \end{gathered} \tag{9}$$

Here O is a column of zeros, n and l the mode numbers of the natural vibrations of the plate, ω_n and ε_l are the natural angular frequencies, U_n and W_l are the column-matrix eigenfunctions of symmetric and antisymmetric vibrations, $M_{0n}, \dots, M_{2N,n}, M_{1l}, \dots, M_{2N+1,l}$ are the moments of different orders due to the appropriate natural vibrations in the cross-section of the plate, and L_s and L_a are the matrix-valued differential operators of the symmetric and antisymmetric vibrations.

We will introduce an abbreviated notation for convolutions of matrix expressions

$$(y, z) = \int_a^b z'(x)\bar{y}(x)dx = \int_a^b \bar{y}'(x)z(x)dx \tag{10}$$

where y and z are column matrices, $\bar{y}$ is a column matrix whose elements are the complex conjugates of those of y, and the prime denotes transposition.

Using (10) and applying Lagrange's formula [4, p. 20] to the matrix equations (9), we obtain

$$(T, L_s U_n) = (\bar{T}' D \partial U_n / \partial x - U_n' D' \partial \bar{T} / \partial x)\big|_a^b + (L_s^* T, U_n)$$

$$(Y, L_a W_l) = (\bar{Y}' Q \partial W_l / \partial x - W_l Q' \partial \bar{Y} / \partial x)\big|_a^b + (L_a^* Y, W_l)$$

It is obvious that, assuming homogeneous boundary conditions at the faces of the plate for the problem and its adjoint, the operators describing the plate vibrations have adjoints [3, 4], i.e.

$$(T, L_s U_n) = (L_s^* T, U_n), \quad (Y, L_a W_l) = (L_a^* Y, W_l)$$

(L_s^*, L_a^* are the adjoint operators).

APPENDIX

The matrix coefficients are

$$D_{j1} = \Sigma \frac{2B_i}{2j-1}\left(\frac{z}{h}\right)^{2j-1}\Bigg|_{z_i}^{z_{i+1}}, \quad R_{j1} = \Sigma \frac{2\rho_i}{2j-1}\left(\frac{z}{h}\right)^{2j-1}\Bigg|_{z_i}^{z_{i+1}}$$

$$D_{js} = -\Sigma \frac{B_i}{(s-1)(2j+2s-3)}\left(\frac{z}{h}\right)^{2j+2s-4}\Bigg|_{z_i}^{z_{i+1}}, \quad K_j = -R_{j1}$$

$$R_{js} = \Sigma\left\{\frac{-\rho_i z}{(s-1)(2j+2s-3)}\left(\frac{z}{h}\right)^{2j+2s-4} - 2A_i\left(I_i\binom{2j-2}{2s-3}(z) + \right.\right.$$

$$\left.\left.+\frac{Rz_{i,2s-3}}{2j-1}\left(\frac{z}{h}\right)^{2j-1}\right)\right\}\Bigg|_{z_i}^{z_{i+1}} + RZ_{m+1,2s-3}\,\Sigma \frac{2A_i}{2j-1}\left(\frac{z}{h}\right)^{2j-1}\Bigg|_{z_i}^{z_{i+1}}$$

$$j = 1, 2, \ldots, N+2, \quad s = 2, 3, \ldots, N+2$$

$$Q_{11} = \tfrac{4}{3}h^3, \quad P_{11} = 0, \quad H_1 = 2RZ_{m+1,0}, \quad P_{1s} = -2RZ_{m+1,2s-4}, \quad Q_{1s} = 0$$

$$P_{j1} = \Sigma\, 2\rho_i\left[\frac{J\binom{2j-3}{i}(z)}{G_i} + \frac{AI_i}{(2j-2)h}\left(\frac{z}{h}\right)^{2j-2}\right]\Bigg|_{z_i}^{z_{i+1}}$$

$$Q_{j1} = \Sigma\, 2\left\{B_i\left(\frac{J\binom{2j-3}{i}(z)}{G_i} + \frac{AI_i}{2jh}\left(\frac{z}{h}\right)^{2j}\right) - A_i J\binom{2j-3}{0}(z)\right\}\Bigg|_{z_i}^{z_{i+1}}$$

$$Q_{js} = -\Sigma \frac{2B_i}{(2s-3)(2j+2s-5)}\left(\frac{z}{h}\right)^{2j+2s-5}\Bigg|_{z_i}^{z_{i+1}}$$

$$P_{js} =$$

$$= \Sigma\, 2\left\{\frac{-\rho_i}{(2s-3)(2j+2s-5)}\left(\frac{z}{h}\right)^{2j+2s-5} - A_i\left(I_i\binom{2j-3}{2s-4}(z) + \frac{RZ_{i,2s-4}}{(2j-2)h}\left(\frac{z}{h}\right)^{2j-2}\right)\right\}\Bigg|_{z_i}^{z_{i+1}}$$

$$H_j = \Sigma\, 2A_i\left(I_i\binom{2j-3}{0}(z) + \frac{RZ_{i0}}{(2j-2)h}\left(\frac{z}{h}\right)^{2j-2}\right)\Bigg|_{z_i}^{z_{i+1}}$$

$$I_i\binom{2s}{j}(z) = \int \frac{z^{2s}}{h^{2s+1}} I_{ij}(z)dz, \quad J\binom{2s+1}{i}(z) = \int \frac{z^{2s+1}}{h^{2s+3}} J_i(z)\,dz$$

$$I_i\binom{2s+1}{j}(z) = \int \frac{z^{2s+1}}{h^{2s+3}} I_{ij}(z)\,dz, \quad s, j = 2, 3, \ldots, N+2$$

Summation is from $i=0$ to $i=m$.

REFERENCES

1. AMBARTSUMYAN S. A., *Theory of Anisotropic Plates.* Nauka, Moscow, 1987.
2. AMBARTSUMYAN S. A., *General Theory of Anisotropic Shells.* Nauka, Moscow, 1974.
3. KORN G. A. and KORN T. M., *Handbook of Mathematics for Scientists and Engineers.* McGraw-Hill, New York, 1968.
4. NAIMARK M. A., *Linear Differential Operators.* Nauka, Moscow, 1969.
5. BOLOTIN V. V. (Ed.), *Vibrations in Engineering. A Handbook in 6 Volumes. Vibrations of Linear Systems*, Vol. 1. Mashinostroyeniye, Moscow, 1978.

Translated by D.L.